Studiu asupra unor probleme

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Problema 1). Sa se rezolve in $\boldsymbol{R}$ ecuatia:

$$\frac{1}{\frac{1}{x}+\frac{1}{x^2}}+\frac{1}{\frac{1}{x^2}+\frac{1}{x^3}}+\frac{1}{\frac{1}{x^3}+\frac{1}{x^4}}+\cdots+\frac{1}{\frac{1}{x^{n-1}}+\frac{1}{x^n}}=\frac{4(x^{n-1}-1)}{x^2-1},$$ unde $x \neq 0, x \neq \pm 1$

<u>Solutie.</u> Fie $S_n=\frac{1}{\frac{1}{x}+\frac{1}{x^2}}+\frac{1}{\frac{1}{x^2}+\frac{1}{x^3}}+\frac{1}{\frac{1}{x^3}+\frac{1}{x^4}}+\cdots+\frac{1}{\frac{1}{x^{n-1}}+\frac{1}{x^n}}=$

$$=\frac{1}{\frac{1}{x}\left(1+\frac{1}{x}\right)}+\frac{1}{\frac{1}{x^2}\left(1+\frac{1}{x}\right)}+\frac{1}{\frac{1}{x^3}\left(1+\frac{1}{x}\right)}+\cdots+\frac{1}{\frac{1}{x^{n-1}}\left(1+\frac{1}{x}\right)}=$$

$$=\frac{1}{\left(1+\frac{1}{x}\right)}\left(x+x^2+x^3+\cdots+x^{n-1}\right)=\frac{x}{x+1}\left(x+x^2+x^3+\cdots+x^{n-1}\right)$$

Se observa ca $x, x^2, x^3, \ldots, x^{n-1}$ este o progresie geometrica cu ratia x

Avem: $x+x^2+x^3+\cdots+x^{n-1}=\frac{x(x^{n-1}-1)}{x-1}$

S_n devine : $S_n=\frac{x}{x+1}\cdot\frac{x(x^{n-1}-1)}{x-1}=\frac{x^2(x^{n-1}-1)}{x^2-1}$

Revenind la ecuatia data avem: $\frac{x^2(x^{n-1}-1)}{x^2-1}=\frac{4(x^{n-1}-1)}{x^2-1}$

Simplificam ecuatia obtinuta cu $\frac{x^{n-1}-1}{x^2-1}$ deoarece $x \neq \pm 1$, si obtinem :

$x^2=4 <=> x=\pm 2$ $\qquad\qquad S=\{\pm 2\}$

<u>**Problema 2).**</u> Aratati ca: $S_n=\sum_{k=2}^{n}\frac{k^3-k-1}{(k+1)!}<\frac{5}{2}$

<u>Solutie</u>. Avem: $k^3-k-1=k(k^2-1)-1=k(k-1)(k+1)-1$

Suma devine:

$$\sum_{k=2}^{n}\frac{k(k-1)(k+1)-1}{(k+1)!}=\sum_{k=2}^{n}\left(\frac{k(k-1)(k+1)}{(k+1)!}-\frac{1}{(k+1)!}\right)= =$$

$$\sum_{k=2}^{n}\left(\frac{k(k-1)(k+1)}{(k-2)!(k-1)k(k+1)}-\frac{1}{(k+1)!}\right)=\sum_{k=2}^{n}\left(\frac{1}{(k-2)!}-\frac{1}{(k+1)!}\right) =$$

$$=\frac{1}{0!}-\frac{1}{3!}+\frac{1}{1!}-\frac{1}{4!}+\frac{1}{2!}-\frac{1}{5!}+\frac{1}{3!}-\frac{1}{6!}+\frac{1}{4!}-\frac{1}{7!}+\frac{1}{5!}-\frac{1}{8!}+\cdots+$$

$$+\frac{1}{(n-5)!}-\frac{1}{(n-2)!}+\frac{1}{(n-4)!}-\frac{1}{(n-1)!}+\frac{1}{(n-3)!}-\frac{1}{n!}+$$

$$+\frac{1}{(n-2)!}-\frac{1}{(n+1)!}=1+1+\frac{1}{2}-\frac{1}{(n-1)!}-\frac{1}{n!}-\frac{1}{(n+1)!}=$$

$$=\frac{5}{2}-\left(\frac{1}{(n-1)!}+\frac{1}{n!}+\frac{1}{(n+1)!}\right)<\frac{5}{2} \text{ doarece } \frac{1}{(n-1)!}+\frac{1}{n!}+\frac{1}{(n+1)!}>0$$

<u>Problema 3).</u> Sa se demonstreze identitatea :

$\frac{\log_a^n-\log_b^n}{\log_{ab}^n}-\log_a^b-\log_b^a$ unde $\quad a,b,n>0 \quad a,b,\ a\cdot b\neq 1$

<u>Solutie</u>. Avem: $\log_a^n=\log_{ab}^n\cdot\log_a^{ab}=\log_{ab}^n\cdot\left(\log_a^a+\log_a^b\right)=$

$$=\log_{ab}^n\left(1+\log_a^b\right) \quad (1)$$

Analog: $\log_b^n-\log_{ab}^n\cdot\log_b^{ab}-\log_{ab}^n\left(\log_b^a+\log_b^b\right)-\log_{ab}^n\left(\log_b^a+1\right) \quad (2)$

Pe baza relatiilor (1) si (2) obtinem:

$$\frac{\log_a^n-\log_b^n}{\log_{ab}^n}=\frac{\log_{ab}^n\left(1+\log_a^b\right)-\log_{ab}^n\left(\log_b^a+1\right)}{\log_{ab}^n}=$$

$$=\frac{\log_{ab}^n\left(1+\log_a^b-\log_b^a-1\right)}{\log_{ab}^n}=\log_a^b-\log_b^a$$

Problema 4). Sa se demonstreze identitatea:

$$C_m^1 + C_m^2 + \cdots + C_m^n =$$

$$- m\left(\frac{1}{m-1} C_{m-1}^{1} + \frac{1}{m-2} C_{m-2}^{2} + \cdots + \frac{1}{m-n} C_{m-n}^{n}\right)$$

Solutie. Pornim de la identitatea: $C_m^p = \frac{m}{m-p} \cdot C_{m-p}^{p}$

Pentru *p* = 1 avem : $C_m^1 = \frac{m}{m-1} C_{m-1}^{1}$

Pentru *p*= 2 avem $C_m^2 = \frac{m}{m-2} C_{m-2}^{2}$

………

Penrtu p=n avem : $C_m^n = \frac{m}{m-n} C_{m-n}^{n}$

Adunam membru cu membru relatiile de mai sus si obtinem:

$$C_m^1 + C_m^2 + \cdots + C_m^n = \frac{m}{m-1} C_{m-1}^{1} + \frac{m}{m-2} C_{m-2}^{2} + \cdots + \frac{m}{m-n} C_{m-n}^{n} =$$

$$= m\left(\frac{1}{m-1} C_{m}^{1}{}_{1} + \frac{1}{m-2} C_{m}^{2}{}_{2} + \cdots + \frac{1}{m-n} C_{m-n}^{n}\right)$$

Problema 5).

Daca $x + \frac{1}{x} = -1$ sa se gaseasca valoarea expresiei $x^{2010} + \frac{1}{x^{2010}}$,

cu $x \in \mathbb{C} \setminus$ **R**

Solutie .

Din $x + \frac{1}{x} = -1$ obtinem x^2+x+1=0 ecuatia care nu are radacini reale

Inmultim ambi membri cu *x-1* si obtinem relatia:

$$(x-1)(x^2 + x + 1) = x^3 - 1 = 0 \; de\ unde\ x^3 = 1$$

Avem: $x^{2010} + \frac{1}{x^{2010}} = (x^3)^{670} + \frac{1}{(x^3)^{270}} = 1 + 1 = 2$

<u>Problema 6).</u> Sa se demonstreze inegalitatea:

$$S = \frac{1}{2!} + \frac{5}{3!} + \frac{11}{4!} + \cdots + \frac{n^2 + n - 1}{(n+1)!} < 2, \qquad n \in N^*$$

<u>Solutie</u> Termenul general al sumei este:

$$\frac{k^2 + k - 1}{(k+1)!} = \frac{k(k+1) - 1}{(k+1)!} = \frac{k(k+1)}{(k+1)!} - \frac{1}{(k+1)!} =$$

$$= \frac{k(k+1)}{(k-1)!/k \cdot (k+1)} - \frac{1}{(k+1)!} = \frac{1}{(k-1)!} - \frac{1}{(k+1)!}$$

Obtinem: $S = \sum_{k=1}^{n} \frac{k^2+k-1}{(k+1)!} = \sum_{k=1}^{n} \left(\frac{1}{(k-1)!} - \frac{1}{(k+1)!}\right) =$

$$= \sum_{k=1}^{n} \frac{1}{(k-1)!} - \sum_{k=1}^{n} \frac{1}{(k+1)!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} -$$

$$-\left(\frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n+1)!}\right) = 1 + \frac{1}{1!} - \frac{1}{n!} - \frac{1}{(n+1)!} =$$

$$= 2 - \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$$

< 2 deoarece $\frac{1}{n!} - \frac{1}{(n+1)!} > 0$