Puteri de matrice pătratice

-Studiu de specialitate-

Adela Terezia Pop, Colegiul Tehnic "Aurel Vlaicu" Baia Mare

1. Se consideră matricele $A=\begin{pmatrix} 1 & 2 & 5 \\ -1 & -2 & -5 \\ 2 & 4 & 10 \end{pmatrix}$ şi $X(a)=I_3+aA$ unde $a\in\mathbf{R}$.

a) Să se demonstreze că $A^2=9\cdot A$ şi că $X(a)\cdot X(b)=X(a+b+9ab)$, $\forall a,b\in\mathbf{R}$.

b) Să se calculeze $(X(a))^n$, unde $n\in\mathbf{N}^*$.

c) Să determine numărul real t, astfel încât $X\left(\frac{-8}{9}\right)\cdot X\left(\frac{-7}{9}\right)\cdot\ldots\cdot X\left(\frac{8}{9}\right)=X(t)$.

d) Să se determine $a\in\mathbf{R}$ astfel încât $(X(a))^{2018}=I_3$.

Rezolvare

a) $A^2=\begin{pmatrix} 1 & 2 & 5 \\ -1 & -2 & -5 \\ 2 & 4 & 10 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 & 5 \\ -1 & -2 & -5 \\ 2 & 4 & 10 \end{pmatrix}=\begin{pmatrix} 9 & 18 & 45 \\ -9 & -18 & -45 \\ 18 & 36 & 90 \end{pmatrix}=9\cdot\begin{pmatrix} 1 & 2 & 5 \\ -1 & -2 & -5 \\ 2 & 4 & 10 \end{pmatrix}=9A$.

$X(a)\cdot X(b)=(I_3+aA)\cdot(I_3+bA)=I_3{}^2+I_3\cdot bA+aA\cdot I_3+abA^2\overset{A^2=9A}{=}I_3+aA+bA+9abA=$
$=I_3+(a+b+9ab)\cdot A=X(a+b+9ab),\forall a,b\in\mathbf{R}$

b) $X(a)\cdot X(b)=X(a+b+9ab)=X\left(9a(b+\frac{1}{9})+b\right)=X\left(9\left(a+\frac{1}{9}\right)\left(b+\frac{1}{9}\right)-\frac{1}{9}\right)$

$(X(a))^2=X(a)\cdot X(a)=X\left(9\left(a+\frac{1}{9}\right)^2-\frac{1}{9}\right)=X\left(\frac{(9a+1)^2-1}{9}\right)$

$P(n):(X(a))^n=X\left(\frac{(9a+1)^n-1}{9}\right)$, $n\in\mathbf{N}^*$.

I) $P(1):X(a)=X\left(\frac{9a+1-1}{9}\right)\Leftrightarrow X(a)=X(a)$, " A"

II) $P(k)\to P(k+1),\forall k\in\mathbf{N}^*$

$P(k):(X(a))^k=X\left(\frac{(9a+1)^k-1}{9}\right)$, presupunem adevărat.

$P(k+1):(X(a))^{k+1}=X\left(\frac{(9a+1)^{k+1}-1}{9}\right)$, trebuie demonstrat.

$$\left(X(a)\right)^{k+1}=\left(X(a)\right)^{k}\cdot X(a)=X\left(\frac{(9a+1)^k-1}{9}\right)\cdot X(a)=$$

$$=X\left(\frac{(9a+1)^k-1}{9}+a+9\cdot\frac{(9a+1)^k-1}{9}\cdot a\right)=X\left(\frac{(9a+1)^k-1+9a+9a\cdot(9a+1)^k-9a}{9}\right)=$$

$$=X\left(\frac{(9a+1)^{k+1}-1}{9}\right)$$

$\Rightarrow$ P(k+1) propoziţie adevărată.

Din $\left.\begin{array}{l}P(1)\text{"}A\text{"}\\ P(k)\to P(k+1),\forall k\in\mathbf{N^*}\end{array}\right\}\overset{M.I.M}{\Rightarrow}$ P(n) propoziţie adevărată, $\forall n\in\mathbf{N^*}$.

Metoda II.

$A^2=9\cdot A$, $A^3=A^2\cdot A=9\cdot A\cdot A=9\cdot 9A=9^2\cdot A$. Prin metoda inducţiei matematice obţinem $A^n=9^{n-1}\cdot A, \forall n\in\mathbf{N^*}$.

Deoarece $I_3\cdot(aA)=(aA)\cdot I_3$ se poate aplica formula binomului lui Newton:

$$\left(X(a)\right)^n=\left(I_3+aA\right)^n=C_n^0\cdot I_3+C_n^1aA+C_n^2a^2A^2+...+C_n^na^nA^{n-1}\overset{A^n=9^{n-1}\cdot A}{=}$$

$$=I_3+\left(C_n^1a+C_n^2a^29+...+C_n^na^n9^{n-1}\right)A=I_3+\frac{\left(C_n^0+C_n^19a+C_n^29^2a^2+...+C_n^n9^na^n-1\right)}{9}A=$$

$$=I_3+\frac{(9a+1)^n-1}{9}\cdot A=X\left(\frac{(9a+1)^n-1}{9}\right).$$

c) Conform punctului a) $X(a)\cdot X\left(\frac{-1}{9}\right)=X\left(a-\frac{1}{9}+9\cdot a\cdot\frac{-1}{9}\right)=X\left(\frac{-1}{9}\right),\forall a\in\mathbf{R}\Rightarrow$

$$X\left(\frac{-8}{9}\right)\cdot X\left(\frac{-7}{9}\right)\cdot...\cdot X\left(\frac{-1}{9}\right)=X\left(\frac{-1}{9}\right).$$

Conform punctului a) $X\left(\frac{-1}{9}\right)\cdot X(b)=X\left(-\frac{1}{9}+b+9\cdot\frac{-1}{9}\cdot b\right)=X\left(\frac{-1}{9}\right),\forall b\in\mathbf{R}\Rightarrow$

$$X\left(\frac{-1}{9}\right)\cdot X\left(\frac{0}{9}\right)\cdot...\cdot X\left(\frac{8}{9}\right)=X\left(\frac{-1}{9}\right).$$

Relaţia din enunţ $X\left(\frac{-8}{9}\right)\cdot X\left(\frac{-7}{9}\right)\cdot...\cdot X\left(\frac{8}{9}\right)=X(t)\Leftrightarrow X\left(\frac{-1}{9}\right)=X(t)\Leftrightarrow t=\frac{-1}{9}$.

d) $\left(X(a)\right)^{2018}=I_3\Leftrightarrow X\left(\frac{(9a+1)^{2018}-1}{9}\right)=X(0)\Leftrightarrow(9a+1)^{2018}=1\overset{a\in\mathbf{R}}{\Leftrightarrow}a\in\left\{\frac{-2}{9},0\right\}$.

2. Se consideră matricea $A(x)=\begin{pmatrix}\cos x+\sin x & 2\sin x\\ -\sin x & \cos x-\sin x\end{pmatrix}$, unde $x\in\mathbf{R}$.

a) Să se demonstreze că $A(x)\cdot A(y)=A(x+y),\ \forall x,y\in\mathbf{R}$.

b) Să se calculeze $A^n(x)$, unde $n\in\mathbf{N^*}$.

c) Dacă $B=\begin{pmatrix}1+\sqrt{3} & 2\sqrt{3}\\ -\sqrt{3} & 1-\sqrt{3}\end{pmatrix}$, să se calculeze B^n, unde $n\in\mathbf{N^*}$.

d) Dacă $C=\begin{pmatrix}\frac{\sqrt{3}+1}{2} & 1\\ -\frac{1}{2} & \frac{\sqrt{3}-1}{2}\end{pmatrix}$, să se determine $n\in\mathbf{N}^*$, astfel încât $C^n=I_2$.

Rezolvare a)

$$A(x)\cdot A(y)=\begin{pmatrix}\cos x+\sin x & 2\sin x\\ -\sin x & \cos x-\sin x\end{pmatrix}\cdot\begin{pmatrix}\cos y+\sin y & 2\sin y\\ -\sin y & \cos y-\sin y\end{pmatrix}=$$

$$=\begin{pmatrix}\cos(x+y)+\sin(x+y) & 2\sin(x+y)\\ -\sin(x+y) & \cos(x+y)-\sin(x+y)\end{pmatrix}=A(x+y),\forall x,y\in\mathbf{R}.$$

b) $A^2(x)=A(x)\cdot A(x)=A(x+x)=A(2x)$.

$A^3(x)=A^2(x)\cdot A(x)=A(2x)\cdot A(x)=A(2x+x)=A(3x)$.

$P(n): A^n(x)=A(nx)$, $n\in\mathbf{N}^*$

I) $P(1): A(x)=A(x)$ propoziţie adevărată

II) $P(k)\to P(k+1),\forall k\in\mathbf{N}^*$

$P(k): A^k(x)=A(kx)$ presupunem adevărat

$P(k+1): A^{k+1}(x)=A((k+1)\cdot x)$ trebuie demonstrat.

$A^{k+1}(x)=A^k(x)\cdot A(x)=A(kx)\cdot A(x)=A(kx+x)=A((k+1)\cdot x)$ $\Rightarrow$P(k+1) propoziţie adevărată.

Din $\left.\begin{matrix}P(1)\text{"A"}\\ P(k)\to P(k+1),\forall k\in\mathbf{N}^*\end{matrix}\right\}\overset{M.I.M}{\Rightarrow}$ P(n) propoziţie adevărată, $\forall n\in\mathbf{N}^*\Rightarrow$

$$A^n(x)=A(nx)=\begin{pmatrix}\cos nx+\sin nx & 2\sin nx\\ -\sin nx & \cos nx-\sin nx\end{pmatrix}.$$

c) $$B=\begin{pmatrix}1+\sqrt{3} & 2\sqrt{3}\\ -\sqrt{3} & 1-\sqrt{3}\end{pmatrix}=2\cdot\begin{pmatrix}\frac{1}{2}+\frac{\sqrt{3}}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}-\frac{\sqrt{3}}{2}\end{pmatrix}=2\cdot\begin{pmatrix}\cos\frac{\pi}{3}+\sin\frac{\pi}{3} & 2\sin\frac{\pi}{3}\\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3}-\sin\frac{\pi}{3}\end{pmatrix}\Rightarrow$$

$$B^n=2^n\cdot\begin{pmatrix}\cos\frac{\pi}{3}+\sin\frac{\pi}{3} & 2\sin\frac{\pi}{3}\\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3}-\sin\frac{\pi}{3}\end{pmatrix}^n\overset{b)}{=}2^n\cdot\begin{pmatrix}\cos\frac{n\pi}{3}+\sin\frac{n\pi}{3} & 2\sin\frac{n\pi}{3}\\ -\sin\frac{n\pi}{3} & \cos\frac{n\pi}{3}-\sin\frac{n\pi}{3}\end{pmatrix}.$$

d) $$C=\begin{pmatrix}\frac{\sqrt{3}+1}{2} & 1\\ -\frac{1}{2} & \frac{\sqrt{3}-1}{2}\end{pmatrix}=\begin{pmatrix}\cos\frac{\pi}{6}+\sin\frac{\pi}{6} & 2\sin\frac{\pi}{6}\\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6}+\sin\frac{\pi}{6}\end{pmatrix}\Rightarrow$$

$$C^n=\begin{pmatrix}\cos\frac{\pi}{6}+\sin\frac{\pi}{6} & 2\sin\frac{\pi}{6}\\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6}+\sin\frac{\pi}{6}\end{pmatrix}^n=\begin{pmatrix}\cos\frac{n\pi}{6}+\sin\frac{n\pi}{6} & 2\sin\frac{n\pi}{6}\\ -\sin\frac{n\pi}{6} & \cos\frac{n\pi}{6}+\sin\frac{n\pi}{6}\end{pmatrix}.$$

$$C^n = I_2 \Leftrightarrow \begin{cases} \sin\frac{n\pi}{6} = 0 \\ \cos\frac{n\pi}{6} = 1 \end{cases} \overset{n\in\mathbf{N}^*}{\Leftrightarrow} \frac{n\pi}{6} \in \{2k\pi \mid k\in\mathbf{N}^*\} \Leftrightarrow n \vdots 12$$

3. Fie $A = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$. Să se calculeze A^n, unde $n \in \mathbf{N}^*$.

Rezolvare.

$$A^2 = A\cdot A = \begin{pmatrix} 3 & 4 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{pmatrix},\ A^3 = A^2\cdot A = \begin{pmatrix} 0 & 0 & 7 & 10 \\ 0 & 0 & 5 & 7 \\ 7 & 10 & 0 & 0 \\ 5 & 7 & 0 & 0 \end{pmatrix}.$$

Fie $A^{2n+1} = \begin{pmatrix} 0 & 0 & a_n & 2b_n \\ 0 & 0 & b_n & a_n \\ a_n & 2b_n & 0 & 0 \\ b_n & a_n & 0 & 0 \end{pmatrix}$, unde $n \in \mathbf{N}$. Din egalitatea $A^{2n+3} = A^{2n+1}\cdot A^2, \forall n \in \mathbf{N}$

obţinem relaţiile de recurenţă $\begin{cases} a_{n+1} = 3a_n + 4b_n, \forall n \in \mathbf{N}, (1) \\ b_{n+1} = 2a_n + 3b_n, \forall n \in \mathbf{N}, (2) \end{cases}$

Din (1) obţinem $b_n = \frac{a_{n+1} - 3a_n}{4}, \forall n \in \mathbf{N}^*$ care substituit în relaţia (2) conduce la $a_{n+2} - 6a_{n+1} + a_n = 0, \forall n \in \mathbf{N}^*$. Folosind ecuaţia caracteristică relaţiei de recurenţă şi condiţiile iniţiale $a_0 = 1, a_1 = 7$ obţinem $a_n = \frac{(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}}{2}$,

$b_n = \frac{\sqrt{2}}{4}\cdot\left[(1+\sqrt{2})^{2n+1} - (1-\sqrt{2})^{2n+1}\right]$.

Prin urmare $A^{2n+1} = \begin{pmatrix} 0 & 0 & a_n & 2b_n \\ 0 & 0 & b_n & a_n \\ a_n & 2b_n & 0 & 0 \\ b_n & a_n & 0 & 0 \end{pmatrix}$, unde $a_n = \frac{(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}}{2}$,

$b_n = \frac{\sqrt{2}}{4}\cdot\left[(1+\sqrt{2})^{2n+1} - (1-\sqrt{2})^{2n+1}\right],\ \forall n \in \mathbf{N}$.

$$A^{2n}=A^{2n-1}\cdot A^2=\begin{pmatrix}0&0&a_{n-1}&2b_{n-1}\\0&0&b_{n-1}&a_{n-1}\\a_{n-1}&2b_{n-1}&0&0\\b_{n-1}&a_{n-1}&0&0\end{pmatrix}\cdot\begin{pmatrix}0&0&1&2\\0&0&1&1\\1&2&0&0\\1&1&0&0\end{pmatrix}=$$

$$=\begin{pmatrix}a_{n-1}+2b_{n-1}&2a_{n-1}+2b_{n-1}&0&0\\a_{n-1}+b_{n-1}&a_{n-1}+2b_{n-1}&0&0\\0&0&a_{n-1}+2b_{n-1}&2a_{n-1}+2b_{n-1}\\0&0&a_{n-1}+b_{n-1}&a_{n-1}+2b_{n-1}\end{pmatrix}=$$

$$=\begin{pmatrix}x_n&2y_n&0&0\\y_n&x_n&0&0\\0&0&x_n&2y_n\\0&0&y_n&x_n\end{pmatrix}, unde\ x_n=\frac{\left(1+\sqrt{2}\right)^{2n}+\left(1-\sqrt{2}\right)^{2n}}{2},\ y_n=\frac{\sqrt{2}}{4}\cdot\left[\left(1+\sqrt{2}\right)^{2n}-\left(1-\sqrt{2}\right)^{2n}\right],$$

$\forall n\in\mathbf{N}^*$

4. O matrice $A\in M_n(\mathbf{C})$ se numeşte ***involutivă*** dacă: $A^2=I_n$;

O matrice $B\in M_n(\mathbf{C})$ se numeşte ***idempotentă*** dacă: $B^2=B$.

Să se arate că:

a) Dacă A este involutivă atunci $\frac{1}{2}(A+I_n)$ este idempotentă.

b) dacă B este indempotentă atunci $2B-I_n$ este involutivă.

Rezolvare:

a) Din $A^2=I_n$ obţinem

$$\left[\frac{1}{2}(A+I_n)\right]^2=\frac{1}{4}\left(A^2+2A+I_n\right)=\frac{1}{4}\left(I_n+2A+I_n\right)=\frac{1}{2}(A+I_n)\Rightarrow\frac{1}{2}(A+I_n)\text{ este idempotentă.}$$

b) Din $B^2=B$ obţinem

$(2B-I_n)^2=4B^2-4B+I_n=4B-4B+I_n=I_n\Rightarrow 2B-I_n$ este involutivă.

5. Se consideră matricea $A=\begin{pmatrix}1&1&1\\1&\varepsilon&\varepsilon^2\\1&\varepsilon^2&\varepsilon\end{pmatrix}$, unde ε este o rădăcină a ecuaţiei $x^2+x+1=0$. Să se calculeze A^n, unde $n\in\mathbf{N}^*$.

Rezolvare.

ε este o rădăcină a ecuaţiei $x^2+x+1=0\Rightarrow\varepsilon^2+\varepsilon+1=0$.

$\varepsilon^3-1=(\varepsilon-1)(\varepsilon^2+\varepsilon+1)=0\Rightarrow\varepsilon^3=1$.

$$A^2=\begin{pmatrix}1 & 1 & 1\\ 1 & \varepsilon & \varepsilon^2\\ 1 & \varepsilon^2 & \varepsilon\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 1\\ 1 & \varepsilon & \varepsilon^2\\ 1 & \varepsilon^2 & \varepsilon\end{pmatrix}=\begin{pmatrix}3 & 0 & 0\\ 0 & 0 & 3\\ 0 & 3 & 0\end{pmatrix}.$$

$$A^3=A^2\cdot A=\begin{pmatrix}3 & 0 & 0\\ 0 & 0 & 3\\ 0 & 3 & 0\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 1\\ 1 & \varepsilon & \varepsilon^2\\ 1 & \varepsilon^2 & \varepsilon\end{pmatrix}=\begin{pmatrix}3 & 3 & 3\\ 3 & 3\varepsilon^2 & 3\varepsilon\\ 3 & 3\varepsilon & 3\varepsilon^2\end{pmatrix}.$$

$$A^4=A^3\cdot A=\begin{pmatrix}3 & 3 & 3\\ 3 & 3\varepsilon^2 & 3\varepsilon\\ 3 & 3\varepsilon & 3\varepsilon^2\end{pmatrix}\cdot\begin{pmatrix}1 & 1 & 1\\ 1 & \varepsilon & \varepsilon^2\\ 1 & \varepsilon^2 & \varepsilon\end{pmatrix}=\begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}=9\cdot I_3.$$

$$A^{4k}=\left(A^4\right)^k=\left(9I_3\right)^k=9^k\cdot I_3,\ \forall k\in\mathbf{N}\Rightarrow A^n=\begin{cases}9^k\cdot I_3, n=4k\\ 9^k\cdot A, n=4k+1\\ 9^k\cdot A^2, n=4k+2\\ 9^k\cdot A^3, n=4k+3\end{cases},\ k\in\mathbf{N}.$$

6. Se consideră matricea $A=\begin{pmatrix}a & b\\ b & a\end{pmatrix}$, cu $a,b\in\mathbf{R}$.

a) Să se arate că $\forall n\in\mathbf{N}^*$, $A^n=\begin{pmatrix}x_n & y_n\\ y_n & x_n\end{pmatrix}$, unde $x_n=\dfrac{(a+b)^n+(a-b)^n}{2}$,

$y_n=\dfrac{(a+b)^n-(a-b)^n}{2}$.

b) Să se rezolve în mulţimea $M_2(\mathbf{R})$ ecuaţia $X^3=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$.

Rezolvare.

$P(n): A^n=\begin{pmatrix}x_n & y_n\\ y_n & x_n\end{pmatrix}$, unde $x_n=\dfrac{(a+b)^n+(a-b)^n}{2}$, $y_n=\dfrac{(a+b)^n-(a-b)^n}{2}$.

I) $P(1): A=\begin{pmatrix}\dfrac{a+b+a-b}{2} & \dfrac{a+b-(a-b)}{2}\\ \dfrac{a+b-(a-b)}{2} & \dfrac{a+b+a-b}{2}\end{pmatrix}\Leftrightarrow A=\begin{pmatrix}a & b\\ b & a\end{pmatrix}$ propoziţie adevărată.

II) $P(k)\to P(k+1), \forall k\in\mathbf{N}^*$

$P(k): A^k=\begin{pmatrix}x_k & y_k\\ y_k & x_k\end{pmatrix}$, unde $x_k=\dfrac{(a+b)^k+(a-b)^k}{2}$, $y_k=\dfrac{(a+b)^k-(a-b)^k}{2}$ presupunem adevărat.

$P(k+1): A^{k+1} = \begin{pmatrix} x_{k+1} & y_{k+1} \\ y_{k+1} & x_{k+1} \end{pmatrix}$, unde $x_{k+1} = \dfrac{(a+b)^{k+1} + (a-b)^{k+1}}{2}$, $y_{k+1} = \dfrac{(a+b)^{k+1} - (a-b)^{k+1}}{2}$

trebuie demonstrat.

$$A^{k+1} = A^k \cdot A = \begin{pmatrix} x_k & y_k \\ y_k & x_k \end{pmatrix} \cdot \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a \cdot x_k + b \cdot y_k & b \cdot x_k + a \cdot y_k \\ b \cdot x_k + a \cdot y_k & a \cdot x_k + b \cdot y_k \end{pmatrix} \Rightarrow$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} a \cdot \dfrac{(a+b)^k + (a-b)^k}{2} + b \cdot \dfrac{(a+b)^k - (a-b)^k}{2} & b \cdot \dfrac{(a+b)^k + (a-b)^k}{2} + a \cdot \dfrac{(a+b)^k - (a-b)^k}{2} \\ b \cdot \dfrac{(a+b)^k + (a-b)^k}{2} + a \cdot \dfrac{(a+b)^k - (a-b)^k}{2} & a \cdot \dfrac{(a+b)^k + (a-b)^k}{2} + b \cdot \dfrac{(a+b)^k - (a-b)^k}{2} \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} \dfrac{(a+b)^{k+1} + (a-b)^{k+1}}{2} & \dfrac{(a+b)^{k+1} - (a-b)^{k+1}}{2} \\ \dfrac{(a+b)^{k+1} - (a-b)^{k+1}}{2} & \dfrac{(a+b)^{k+1} + (a-b)^{k+1}}{2} \end{pmatrix} \Rightarrow P(k+1) \text{ propoziţie adevărată.}$$

Din $\left. \begin{array}{l} P(1)\text{"}A\text{"} \\ P(k) \to P(k+1), \forall k \in \mathbf{N}^* \end{array} \right\} \overset{M.I.M}{\Rightarrow}$ P(n) propoziţie adevărată, $\forall n \in \mathbf{N}^*$.

b) Dacă $X^3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ atunci $X^4 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot X$ şi $X^4 = X \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, prin urmare

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot X = X \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Fie $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Obţinem:

$$\begin{cases} 2a + c = 2a + b \\ 2b + d = a + 2b \\ a + 2c = 2c + d \\ b + 2d = c + 2d \end{cases} \Leftrightarrow \begin{cases} d = a \\ c = b \end{cases} \Rightarrow X = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$\overset{conform\, a)}{\Rightarrow} X^3 = \begin{pmatrix} \dfrac{(a+b)^3 + (a-b)^3}{2} & \dfrac{(a+b)^3 - (a-b)^3}{2} \\ \dfrac{(a+b)^3 - (a-b)^3}{2} & \dfrac{(a+b)^3 + (a-b)^3}{2} \end{pmatrix}.$$

$$X^3 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dfrac{(a+b)^3 + (a-b)^3}{2} & \dfrac{(a+b)^3 - (a-b)^3}{2} \\ \dfrac{(a+b)^3 - (a-b)^3}{2} & \dfrac{(a+b)^3 + (a-b)^3}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} (a+b)^3 + (a-b)^3 = 4 \\ (a+b)^3 - (a-b)^3 = 2 \end{cases} \Leftrightarrow \begin{cases} (a+b)^3 = 3 \\ (a-b)^3 = 1 \end{cases} \overset{a,b \in \mathbf{R}}{\Leftrightarrow} \begin{cases} a + b = \sqrt[3]{3} \\ a - b = 1 \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{\sqrt[3]{3} + 1}{2} \\ b = \dfrac{\sqrt[3]{3} - 1}{2} \end{cases}, \; X = \begin{pmatrix} \dfrac{\sqrt[3]{3} + 1}{2} & \dfrac{\sqrt[3]{3} - 1}{2} \\ \dfrac{\sqrt[3]{3} - 1}{2} & \dfrac{\sqrt[3]{3} + 1}{2} \end{pmatrix}.$$